Cone fields and topological sampling in manifolds with bounded curvature

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Abstract

Often noisy point clouds are given as an approximation of a particular compact set of interest. A finite point cloud is a compact set. This paper proves a reconstruction theorem which gives a sufficient condition, as a bound on the Hausdorff distance between two compact sets, for when certain offsets of these two sets are homotopic in terms of the absence of μ-critical points in an annular region. Since an offset of a set deformation retracts to the set itself provided that there are no critical points of the distance function nearby, we can use this theorem to show when the offset of a point cloud is homotopy equivalent to the set it is sampled from. The ambient space can be any Riemannian manifold but we focus on ambient manifolds which have nowhere negative curvature. In the process, we prove stability theorems for μ-critical points when the ambient space is a manifold.

1   Introduction

In modern science and engineering a common problem is understanding some shape from a point cloud sampled from that shape. This point cloud should be thought of as some finite number of (potentially) noisy samples. Topology and geometry are considered very natural tools in such data analysis (see e.g. [14] and [4]). One reason is because topological invariants are often more stable under noise. We will want to understand the homotopy type of a set - two objects are homotopy equivalent if there is a way to continuously deform one object into another.

Often we wish to know the extent to which we can, and how to, reconstruct shapes from noisy point clouds. Naturally the more restrictions on the space the easier it is to reconstruct. The first area of focus was the study of surfaces in $\mathbb{R}^3$, motivated by problems such as medical imaging, visualization and reverse engineering of physical objects. Algorithms with theoretical guaranties exist for smooth closed surfaces with sufficient dense samples. The sampling conditions are based on the concept of ϵ-sampling. A point cloud of A is an ϵ-sample if for every point $p \in A$ there is some sample point at distance at most ϵlfs(p) away, where lfs(p) (called the local feature size at p) is the distance from p to the medial axis of A. The Cocone Algorithm produces a homeomorphic set from any 0.06-sampling[1] of a smooth closed surface. This process has been extended to smooth surfaces with boundaries [9]. However given an arbitrary compact set, the best we can hope for is to find some nearby set that is homotopy equivalent. For instance from a point cloud we can not tell apart the original set and the same set with a slight thickening in places.

One of the simplest methods of reconstruction is to use the offset of a sampling. Given a set K, the r-offset of K, denoted K_r, is defined to be $\{x \in \mathcal{M} : d(x, K) \leq r\}$. This is topologically the same as taking the α-shape of data points [12] or taking the Cech complex [6]. This leads to the problem of finding theoretical guarantees as to when an offset of a sampling has the same topology (i.e. homotopy type) as the underlying set. In other words we want to find conditions on a point

cloud S of a compact set A so that S_r is homotopy equivalent to A. We will in fact find sufficient conditions for S_r to deformation retract to A. Clearly this will only work if the point cloud is sufficiently close. Usually "sufficiently close" is interpreted as a bound on the Hausdorff distance between A and S (the Hausdorff distance between A and S, denoted $d_H(A, S)$, is the smallest $r \geq 0$ such that $S \subset A_r$ and $A \subset S_r$). Much earlier theory assumes that this Hausdorff distance is less than some measure of geometrical or topological feature size of the shape and show the output is correct. We now survey some of this development.

The medial axis of a compact set A is the set of points p in the ambient space for which there is more than one point in A which is closest to p. The reach of A is the minimum distance between points in A and points in the medial axis of A. It can be thought of as the minimum local feature size. Sampling conditions based on reach include those found in [20] which consider smooth manifolds in $\mathbb{R}^n$. Smooth submanifolds have positive reach but a wedge, for instance, does not. To deal with a larger class of sets Chazal, Cohen-Steinerm and Lieutier in [7] introduced the notion of μ-reach. A point is μ-critical when the norm of the gradient of the distance function at that point is less than or equal to μ (a geometric description is expanded on later - see section 3). In particular 0-critical points are critical points of the distance function and every point on the medial axis is a μ-critical point for some $\mu < 1$. The μ-reach of a set is the supremum of $r > 0$ such that the r offset such not contain any μ-critical points. In [7] and [2] sampling conditions are given in terms of the μ-reach.

Instead of making our bounds in terms of μ-reach, we will only require the absence of μ-critical points in an annular region of A along with a bound on the weak feature size. The weak feature size is the infimum of the positive critical values of the distance function from A. This has several advantages. Firstly, it can mean a significant improvement on the bounds such as in the case of "hairy" objects. Secondly, it can be applied to a larger class of compact sets. Instead of positive μ-reach, all that is necessary for our reconstruction process to work is for A to have positive weak feature size. Compact subsets with a cusp or including two tangential spheres are examples of compact sets with zero μ-reach for all $\mu > 0$ but these still have positive weak feature size. Every semi-algebraic set has positive weak feature size [13]. This follows from the fact that the distance function from a semi-algebraic set has only finitely many critical values.
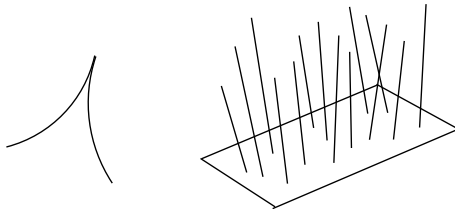
Example of a cusp and a hairy object

One limitation to any of these reconstruction theorems is the requirement of knowing geometric properties of the unknown object we are trying to reconstruct. A shift in perspective can overcome this limitation by considering the geometric properties of the point cloud, which we do know, and can hence prove sufficient conditions for an offset of the original set to deformation retract to an offset of the point cloud. We know the point cloud and hence we know the μ-critical values of its distance function. Theoretical guarantees are given in [7] for when suitable homotopies exist by considering the μ-reach of an offset of the point cloud. Unfortunately there is usually a significant number of small critical values of the distance function to the point cloud. This means the starting offset beyond which μ-reach is considered is significant. Our approach which only considers the existence of μ-critical points in a annular region thus gains a significant advantage.

We note that previous reconstruction theories have been restricted to the case where the ambient space is Euclidean. Another contribution of this paper is to allow the ambient space to be any

manifold whose curvature is bounded from below, thus answering an open question asked in [7]. Although we focus on the important case of non-negatively curved manifolds we explore a paradigm of reconstruction which can be applied to manifolds with curvature bounded from below by some $\kappa < 0$ with analogous, albeit messier, results. Examples of manifolds with nowhere negative curvature are Stiefel and Grassmannian manifolds. These examples are important because there are many applications where data naturally lies on these manifolds such as in dynamic textures [10], face recognition [5], gait recognition [3] and affine shape analysis and image analysis [21].

Even when restricted to the case where we use μ-reach in Euclidean space we still improve on the previous results whenever $\mu \leq 0.945$. Our main theorem expressed as a sampling condition is as follows.

Theorem. *Let $\mu \in (0,1)$, $r > 0$. Let A be a compact subset of a smooth manifold $\mathcal{M}$ with nowhere negative curvature such that the injectivity radius of every point in $\mathcal{M}$ is greater than r and A_r is homotopic to A. Let S be a (potentially noisy) finite point cloud of A (i.e. a finite set of points). Suppose that either*

(i) *there are no μ-critical points of the distance function from A in $\{x \in \mathcal{M} : d(x, A) \in [a, b]\}$*

   or

(ii) *there are no μ-critical points of the distance function from S in $\{x \in \mathcal{M} : d(x, S) \in [a, b]\}$.*

Then S_r is homotopic to A whenever $d_H(S, A) \leq \min\left\{r - a, \frac{b\mu - r\mu}{4 - \mu}, \frac{\mu^2 r}{4 + \mu^2}\right\}$.

Note that a sufficient condition for A_a to defamation retract to A is that the weak feature size of A is at least a [15].

Our approach is to show the existence of a Lipschitz unit vector field whose integral flow induces a deformation retraction from S_r to $A_{r-\delta}$ or from A_r to $S_{r-\delta}$. For the flow of a vector field to have a certain desirable property, we consider sufficient local conditions on potential unit tangent vectors at each point. To this end we define cone fields; cone fields are generalizations of not necessarily continuous unit vector fields where we attach a closed ball in the unit tangent sphere, a "cone", at each point in the manifold. To prove the existence of useful cone fields we need to generalize previous stability results about μ-critical points. A stability result of μ-critical points when the ambient space is Euclidean is proved in [7]. We prove a generalization of this result for when the curvature of the ambient space is bounded from below. It is worth observing that although μ-reach is not stable under Hausdorff distance[1] we do have some stability of an absence of μ-critical points within of annular regions. A quantitative analysis of this phenomenon leads to the theorem.

2 Notation and cone fields

We first need to establish some notation. Throughout $(\mathcal{M}, g)$ is a smooth n-dimensional manifold without boundary and $K \subset \mathcal{M}$ is a compact subset. We will denote the closed ball of radius r and centre x by $B(x, r)$. Let d_K denote the distance function from K. We will use for convenience: $K_\delta := d_K^{-1}[0, \delta]$ and $K_{[a,b]} := d_K^{-1}[a, b]$. The Hausdorff distance between two compact sets K and L is denoted $d_H(K, L)$ and is defined by

$$d_H(K, L) := \max\{\sup_{x \in K} d_L(x), \sup_{y \in L} d_K(y)\}.$$

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[1]In particular, for any compact set K and any bound on Hausdorff distance $\delta > 0$ there is a compact set L with zero μ-reach such that $d_H(K, L) < \delta$

Alternatively it is the smallest $r \geq 0$ such that $K \subset L_r$ and $L \subset K_r$.

Let K° denote the interior of K. Given a metric space X, let $\mathcal{K}(X)$ denote the metric space of compact subsets of X with the metric determined by Hausdorff distance.

We want to recall some differential geometry. A useful reference is [17]. The unit tangent bundle of manifold a $(\mathcal{M}, g)$, denoted by $UT\mathcal{M}$, is the unit sphere bundle for the tangent bundle $T\mathcal{M}$. It is a fiber bundle over $\mathcal{M}$ whose fiber at each point is the unit sphere in the tangent plane;

$$UT\mathcal{M} := \coprod_{x \in M} \{v \in \mathrm{T}_x(M) : g_x(v, v) = 1\},$$

where $T_x \mathcal{M}$ denotes the tangent space to $\mathcal{M}$ at x. Thus, elements of $UT\mathcal{M}$ are pairs (x, v), where x is some point of the manifold and v is some tangent direction (of unit length) to the manifold at x.

The exponential map at x is a map from the tangent space $T_x\mathcal{M}$ to $\mathcal{M}$. For any $v \in T_x\mathcal{M}$, a tangent vector to $\mathcal{M}$ at x, there is a unique geodesic γ_v satisfying $\gamma_v(0) = x$ with initial tangent vector $\gamma_v'(0) = v$. The exponential map at x is defined by $\exp_x(v) = \gamma_v(1)$. The injectivity radius at a point x is the radiusof the largest ball on which the exponential map at x is a diffeomorphism. Normal coordinates at a point x are a local coordinate system in a neighborhood of x obtained by applying the exponential map to the tangent space at x.

Consider the $(n-1)$-dimensional unit sphere, S^{n-1}, lying inside $\mathbb{R}^n$ with a metric induced from this embedding. Denote by $C(w, \beta)$ the closed ball in S^{n-1} centered at v with radius β. We can view $C(w, \beta)$ as the intersection of S^{n-1} with a particular infinite cone;

$$C(w, \beta) = S^{n-1} \cap \{v \in \mathbb{R}^n \backslash \{0\} : \angle(v, w) \leq \beta\}.$$

Here all the angles are in radians. We say $C(w, \beta)$ is *acute* if β is acute.

Denote by F the fibre bundle over $\mathcal{M}$ where each fibre over $x \in \mathcal{M}$ is the space of non-empty closed balls in the unit tangent sphere at x. F has a natural metric induced from the Hausdorff metric on compact subsets of $UT\mathcal{M}$.

A *cone field* over a subset $U \subset \mathcal{M}$ is a section of F restricted to U. A cone field is continuous if it is continuous as a section. We can consider a cone field as a section from $U \subset \mathcal{M}$ into $\mathcal{K}(UT\mathcal{M})$, the space of compact subsets of $UT\mathcal{M}$ with Hausdorff metric, and it is continuous if this map is continuous. As a set, we can write cone fields as

$$\{(x, C(w_x, \beta_x)) : w_x \in UT_x\mathcal{M}, \beta_x \in [0, \pi]\}.$$

Given two cone fields V and W we will write $W \geq V$ to mean that for any point x the cone at x in V is contained in the cone at x in W.

Let K be a compact subset of $\mathbb{R}^n$ not containing the origin. We say $C(w, \beta)$ is a *minimal spanning cone* for K if $\hat{K} := \{\frac{y}{|y|} : y \in K\} \subset C(w, \beta)$ and $\hat{K} \subset C(w', \beta')$ implies $\beta \leq \beta'$.

Lemma 2.1. *Let $K \subset \mathbb{R}^n$ be a non-empty compact subset set such that 0 is not in the convex hull of K. Then there exists a unique minimal spanning cone of K which is acute. Furthermore the map from compact subsets of $\mathbb{R}^d$ which don't have the origin in their convex hull to their minimal spanning cone is locally Lipschitz. If we restrict our attention to compact subsets whose distance to the origin is bounded from below by r we further have that the map sending each compact subset to its minimal spanning cone is Lipschitz with Lipschitz constant bounded from above by $1/r$.*

Proof. From the compactness of $\hat{K}$ we know that, given any $v \in S^{n-1}$, there is some angle β_v such that $\hat{K} \subset C(v, \beta_v)$ and if $\hat{K} \subset C(v, \beta_v')$ then $\beta_v \leq \beta_v'$. Observe that for many choices of v here the

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corresponding β_v is not acute. Set $\beta := \inf\{\beta_v : v \in S^{n-1}\}$. The map $v \mapsto \beta_v$ is clearly continuous and hence the infimum is achieved. Any $C(v, \beta_v)$ where $\beta_v = \beta$ is by definition a minimal spanning cone for K.

Since 0 is not in the convex hull of K, there is some set of coordinates such that K lies in the upper half plane. This implies that is some w such that $\hat{K} \subset C(w, \pi/2)^\circ$. This implies that the corresponding angle β_w is acute. This means that any minimal spanning cone of K is acute.

Thus all it remains to show is that there is only one $w \in S^{n-1}$ such that $\beta_w = \beta$. Suppose that there are two distinct such vectors w_1 and w_2. Now $\hat{K} \subset C(w_1, \beta)$ and $\hat{K} \subset C(w_2, \beta)$ and hence $\hat{K} \subset C(w_1, \beta) \cap C(w_2, \beta)$. The angle between w_1 and w_2 must be less that $2\beta < \pi$ as otherwise $C(w_1, \beta) \cap C(w_2, \beta)$ would be empty. Let $\theta = \angle(w_1, w_2)$. Set $w := (w_1 + w_2)/\sqrt{2 + 2\cos\theta}$ which is a unit vector by construction.

Let $x \in K$ and v the unit vector in the direction of x from 0. We have $\angle(v, w_1) \leq \beta$ and $\angle(v, w_2) \leq \beta$ but since w_1, w_2 and v are all unit vectors we have $\langle v, w_1 \rangle \geq \cos\beta$ and $\langle v, w_2 \rangle \geq \cos\beta$. From the choice of w we have $\langle v, w \rangle \geq 2\cos\beta/\sqrt{2 + 2\cos\theta}$ which implies that $\angle(v, w) =: \alpha_x$ is strictly less than β.

Since K is compact we have $\alpha := \sup\{\alpha_x : x \in K\}$ is attained and $\alpha < \beta$. However this implies that K is contained in a cone with an angle smaller than β which is a contradiction. Thus the choice of centre vector w is unique.

Let K be a compact subset of $\mathbb{R}^n$ whose convex hull does not contain the origin with a corresponding spanning cone C(K). There is some $\delta > 0$ such that $K_{2\delta}$ also does not contain the origin in its convex hull. Let L be a compact subset of $\mathbb{R}^n$ such that $d_H(K, L) < \epsilon < \delta$. Let π be the canonical projection map from $\mathbb{R}^n \backslash 0$ onto S^{n-1}. We have $d_H(\pi(K), \pi(L)) \leq d_H(K, L)/r < \epsilon/r$. where $r = \min\{\|x\| : x \in K_\delta\} > 0$.

Since $L \subset K_\delta$ we have that the convex hull of L does not contain the origin and hence there exists a minimal spanning cone $C(L)$ furthermore $L_\epsilon \subset K_{2\delta}$ and hence L_ϵ also has a minimal spanning cone $C(L_\epsilon)$.

Now $\pi(L) \subset \pi(K_\epsilon) \subset C(\pi(K_\epsilon)) \subset C(\pi(K))_{\epsilon/r}$ implying $C(L) = C(\pi(L)) \subset C(K)_{\epsilon/r}$ and similarly we know $C(K) \subset C(L)_{\epsilon/r}$. Together they imply that $d_H(C(K), C(L)) \leq \epsilon/r$. $\qquad\square$

We can consider vectors inside the cone at a point x. We say a vector field $X := \{(x, v_x); x \in U, v_x \in T_x\mathcal{M}\}$ is *subordinate* to the cone field $W = \{(x, c_x)\}$ if v_x always lies in c_x. We will call a vector field *strictly subordinate* if the vector at each point lies in the interior of the cone.

Given a compact subset $K \subset \mathcal{M}$, a point $x \in \mathcal{M}\backslash K$ and some $R > 0$, consider the compact subset $\exp_x^{-1} K \cap B(0, R)$ of $T_x\mathcal{M}$. The straight line segments from 0 to points in $\exp_x^{-1} K \cap B(0, R)$ are sent under $\exp_x$ to geodesics of length at most R from x to points in K. If 0 is not in the convex hull of $\exp_x^{-1} K \cap B(0, R)$ then by the previous lemma there exists a unique acute minimal spanning cone for $\exp_x^{-1} K \cap B(0, R)$ from 0. Considered as a subset of $UT_x\mathcal{M}$, we call this the *minimal spanning cone* for K from x of length R. If, for some subset $U \in \mathcal{M}$, 0 is not in the convex hull of $\exp_x^{-1} K \cap B(0, R)$ for any $x \in U$ then we can define over U the *minimal cone field* for K with length R by assigning to each x the corresponding minimal spanning cone.

In general, even if a minimal spanning cone field exists over $U \in \mathcal{M}$ it may not be continuous. However we have the following sufficient condition for the minimal cone field to be locally Lipschitz.

Proposition 2.2. *Let $R > 0$, and K, L compact subsets of $\mathcal{M}$ such that the injectivity radius of every point in K is greater than R, $K \subset B(x, R)$ for all $x \in L$, and $K \cap L = \emptyset$. If there exists an acute minimal cone field over L for K of length R then that minimal cone field is Lipschitz.*

Proof. Since K and L are compact and $K \cap L = \emptyset$ there is some $\epsilon > 0$ such that $L_\epsilon \cap K = \emptyset$ and the injectivity radius for every point in K is at least $R + \epsilon$

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For each fixed $y \in K$ we can construct the map

$$\phi_y : L_\epsilon \to \mathcal{K}(T\mathcal{M})$$
$$x \mapsto \{v_x^y\}$$

where v_x^y is the tangent vector to the unique shortest geodesic from x to y with length $d(x, y)$. This is well-defined because of our injectivity radius assumption and because $K \cap L = \emptyset$. The map ϕ_y is smooth. Since L is compact and $L \subset L_\epsilon^\circ$ we can conclude that $\phi_y|_L$ is Lipschitz with some Lipschitz constant α_y.

Set $\alpha := \sup\{\alpha_y : y \in K\}$ which is finite as K is compact and the map $y \mapsto \alpha_y$ is continuous. This means every ϕ_y is Lipschitz with Lipschitz constant bounded from above by α.

Now consider the map

$$\phi : L \to \mathcal{K}(T\mathcal{M})$$
$$x \mapsto \bigcup_{y \in K} v_x^y.$$

Observe that $d_H(\phi_y(x), \phi(x')) \leq d_H(\phi_y(x), \phi_y(x'))$ and $d_H(\phi(x), \phi_y(x')) \leq d_H(\phi_y(x), \phi_y(x'))$ for all $y \in K$. This implies that

$$d_H(\phi(x), \phi(x')) \leq \sup\{d_H(\phi_y(x), \phi_y(x')) : y \in K\} \leq \alpha d(x, x').$$

and hence ϕ is also Lipschitz.

From $d(x, y) > \epsilon$ for all $x \in L$ and $y \in K$ we know that $d(0, \bigcup_{y \in K} v_x^y) > \epsilon$. Recall that in Lemma 2.1 we showed that the map that sends compact subsets whose distance to the origin is bounded from below by ϵ and do not have 0 in there convex hull to their minimal spanning cone is Lipschitz.Since the composition of Lipschitz maps is Lipschitz, we conclude that the minimal spanning cone field is Lipschitz. $\qquad\square$

We can define the complementary cone for all acute cones. The complementary cone of $C(w, \beta)$ is $C(w, \pi/2 - \beta)$. Given a cone field where the cone at each point is acute we can construct the complementary cone field pointwise. From the triangle inequality on the unit tangent sphere we obtain the following useful lemma.

Lemma 2.3. *Let v be a vector in some acute cone C and v' a vector strictly inside the complementary cone. Then $\angle(v, v') < \pi/2$.*

3   Stability of μ-critical points

We want to study the gradient vector fields for distance functions from compact subsets of a general manifold $(\mathcal{M}, g)$. This can be thought of as the obvious generalization of the gradient vector fields for distance functions from compact subsets of Euclidean space (as studied in [7] and [18]).

We call γ a *segment* from $x \notin K$ to K is if γ is a distance achieving path from x to K. If $\mathcal{M}$ is Euclidean then these segments are straight lines. Observe that on an arbitrary manifold there can be more than one segment connecting x to the same $y \in K$. The point x is a called a *critical point* of the distance function from K if, for all non-zero $v \in T_x\mathcal{M}$, there exists a segment γ from x to K such that $\angle(\gamma'(0), v) \leq \pi/2$. Equivalently, if $\Gamma(x) := \{y \in K : d_K(x) = d(x, y)\}$, then x is a critical point if and only if 0 lies in the convex hull of $\exp_x^{-1} \Gamma(x) \cap B(0, d_K(x))$.

We need to construct the gradient vector field so that it vanishes at critical points of the distance function. For all non-critical points we can consider the minimal spanning cone $C(w_x, \beta_x)$ for $\Gamma(x)$ from x of length $d_K(x)$. We set

$$\nabla_K(x) := -\cos(\beta_x) w_x$$

whenever x is not critical. Observe that $\nabla_{K_a}(x) = \nabla_K(x)$ whenever $d_K(x) > a$. For $\mu \in \mathbb{R}$, we called x *μ-critical* if $\|\nabla_K(x)\| \leq \mu$. A point is 0-critical exactly when it is a critical point for the distance function.

It is easy to verify that these definitions agree with those given in [7] when $\mathcal{M}$ is Euclidean.

We will want to prove a generalization of the stability result in [7] where the ambient space is a manifold with curvature bounded from below in a suitable neighborhood of the compact subset under study. By appropriate scaling it is sufficient to consider the cases where the curvature is bounded from below by 1, 0 or -1.

Lemma 3.1. *Let $K \subset \mathcal{M}$ be a compact subset of a manifold $(\mathcal{M}, g)$ for which the curvature on $K_{2\alpha}$ is bounded from below by κ. Let $x \in K_\alpha \backslash K$ be a μ-critical point of d_K. Then for any $y \in K_{2\alpha}$ we have*

$$\cos d_K(y) \geq \cos d(y,x) \cos d_K(x) - \sin d(y,x) \sin d_K(x) \mu \qquad \text{if } \kappa = 1$$
$$d_K(y)^2 \leq d_K(x)^2 + d(x,y)^2 + 2d(y,x)d_K(x)\mu \qquad \text{if } \kappa = 0$$
$$\cosh d_K(y) \leq \cosh d(y,x) \cosh d_K(x) + \sinh d(y,x) \sinh d_K(x) \mu \qquad \text{if } \kappa = -1.$$

Proof. Let $\theta \in (0, \pi/2]$ such that $\cos \theta = \mu$. Let $\Gamma(x) = \{z \in K : d_K(z) = d(z,x)\}$ and set

$$\widehat{\Gamma(x)} := \{z/\|z\| : z \in \exp_x^{-1}(\Gamma(x))\}.$$

Fix $y \in K_{2\alpha}$ and choose $v \in T_x\mathcal{M}$ such that $\exp_x(d(x,y)v) = y$.

We want to show that there is some $z \in K$ and length achieving geodesics γ_x^y and γ_x^z such that $\angle(y,x,z) \leq \pi - \theta$ where $\angle(y,x,z)$ is the angle between γ_x^y and γ_x^z. Suppose not. This means that no point of $\widehat{\Gamma}(x)$ lies in $C(v, \pi - \theta)$. Geometrically this means that $\widehat{\Gamma}(x)$ must lie in $C(-v,\theta)^\circ$ which is the complement of $C(v, \pi - \theta)$ in the sphere (recall that $C(-v,\theta)^\circ$ is the interior of $C(-v,\theta)$).

However this implies that the minimal spanning cone for $\Gamma(x)$ from x of length $d_K(x)$ lies strictly inside $C(-v,\theta)$ and hence $\|\nabla_K(x)\| > \cos\theta = \mu$. This contradicts the assumption that x is a μ-critical point (i.e. $\|\nabla_K(x)\| \leq \mu$). Thus by contradiction, there is some point $z \in K$ and length achieving geodesics γ_x^y and γ_x^z such that $\angle(y,x,z) \leq \pi - \theta$.

Let $\triangle_{x,y,z}$ be the geodesic triangle with γ_x^y and γ_x^z such that $\angle(y,x,z) \leq \pi - \theta$ which must exist. Let $\tilde{\triangle}_{\tilde{x}, \tilde{y}, \tilde{z}}$ be the corresponding triangle in $\mathcal{M}(\kappa)$, the manifold with constant curvature κ, where the length of the sides are preserved. Toponogov's theorem is a triangle comparison theorem which quantifies the assertion that a pair of geodesics emanating from the same point spread apart more slowly in a region of high curvature than they would in a region of low curvature. The details of this theorem and its proof can be found in [8]. By taking the contrapositive of Toponogov's theorem we know $\angle(\tilde{y}, \tilde{x}, \tilde{z}) \leq \angle(y,x,z) \leq \pi - \theta$ and hence $\cos \angle(\tilde{y}, \tilde{x}, \tilde{z}) \geq -\mu$.

We finally substitute $d_K(y) \leq d(\tilde{y}, \tilde{z})$, $d_K(x) = d(\tilde{x}, \tilde{z})$ and $\cos \angle(\tilde{y}, \tilde{x}, \tilde{z}) \geq -\mu$ into the spherical, Euclidean and hyperbolic cosine rules respectively to obtain the desired inequalities. $\qquad \square$

Our stability result will arise from comparing two opposing inequalities - one from the previous lemma alongside one coming from the following lemma which is Lemma 4.1 in [19]. It is easy to check the definition of $\|\nabla_K(x)\|$ coincides with $\|\nabla_x f\|$ when $f = d_K$ and X is a manifold.[2]

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[2] $\|\nabla_x f\|$ is the nonnegative number $\max\{0, \limsup_{y \to x} \frac{f(y) - f(x)}{d(y,x)}\}$. That this is $\|\nabla_K(x)\|$ follows from our geometric construction of $\nabla_K(x)$ and from the cosine rule.

Lemma 3.2 (Lemma 4.1 in [19])**.** *Let X be a metric space. Suppose $f : X \to \mathbb{R}$ is a locally Lipschitz map, $x \in X$, and $f(x) = 0$. For $\mu, r > 0$, assume that the ball $B(x, r))$ is complete and that $\|\nabla_z f\| \geq \mu$ for each z with $d(z, x) < r$ and $f(z) \geq 0$. Then for each $0 < C < \mu$ there is a point $z \in X$ with $d(z, x) \leq r$ and $f(z) = Cr$.*

The following proposition is a generalization of critical point stability theorem in [7] where the ambient space is can now any manifold with non-negative curvature.

Proposition 3.3. *Let K, L be compact subsets of $\mathcal{M}$ with $d_H(K, L) \leq \delta$. Let x be a μ-critical point of d_K. If*

$$C \geq \mu + 2\sqrt{\frac{\delta}{d_K(x)}}$$

and $K_{d_K(x)+4\delta/(C-\mu)}$ has nowhere negative curvature, then there exists a C-critical point y of d_L with $d_L(y) \geq d_L(x)$ and $y \in B(x, 4\delta/(C - \mu))$

Proof. We want to show that there is some y such that $\|\nabla_L(x)\| \leq C$ and $d_L(y) \geq d_L(x)$ and $d(x, y) \leq 4\delta/(C - \mu)$. Suppose not. Then there is some $\tilde{\mu} > C$ such that $\|\nabla_L(x)\| \geq \tilde{\mu}$ whenever $d_L(y) \geq d_L(x)$ and $d(x, y) \leq 4\delta/(C - \mu)$.

If $C \geq \mu + 2\sqrt{\delta/d_K(x)}$ then $d_K(x) - 4\delta/(C - \mu)^2 \geq 0$ and hence we can construct

$$K' := K_{d_K(x)-4\delta/(C-\mu)^2} \text{ and } L' := L_{d_K(x)-4\delta/(C-\mu)^2}.$$

By construction $d_H(K', L') \leq d_H(K, L) \leq \delta$ and $d_{K'}(x) = 4\delta/(C - \mu)^2$. Using $f = d_{L'} - d_{L'}(x)$ and $r = 4\delta/(C - \mu)$ in Lemma 3.2 we know that there exists a point $y \in B(x, 4\delta/(C - \mu))$ such that $f(y) = C4\delta/(C - \mu)$ which means $d_{L'}(y) = d_{L'}(x) + C4\delta/(C - \mu)$. Using $d_H(K', L') \leq \delta$ we can show that

$$d_{K'}(y) \geq d_{L'}(y) - \delta = d_{L'}(x) + C4\delta/(C - \mu) - \delta \geq d_{K'}(x) + C4\delta/(C - \mu) - 2\delta.$$

Since $d_{K'}(x) = 4\delta/(C - \mu)^2$ we conclude that

$$d_{K'}(y) \geq \frac{4\delta + C4\delta(C - \mu) - 2\delta(C - \mu)^2}{(C - \mu)^2}. \tag{1}$$

At the same time, Lemma 3.1 implies that $d_{K'}(y)^2 \leq d_{K'}(x)^2 + d(x, y)^2 + 2d(y, x)d_{K'}(x)\mu$ and hence

$$d_{K'}(y)^2 \leq \frac{16\delta^2 + 16\delta^2(C - \mu)^2 + 32\delta^2\mu(C - \mu)}{(C - \mu)^4} \tag{2}$$

By combining (1) and (2) and performing some algebraic manipulation we obtain

$$1 + (C - \mu)^2 + 2(C - \mu)\mu \geq (1 + C(C - \mu) - (C - \mu)^2/2)^2. \tag{3}$$

However algebraic manipulation of (3) implies $0 \geq (C - \mu)^2/4 + C\mu$ which is a contradiction. $\square$

One of the problems with working with the μ-reach is that it is not stable under Hausdorff distance. Indeed by the creation of an arbitrarily small cusp we know that for any compact subset K, and any $\delta > 0$, there exists some compact subset L with $d_H(K, L) < \delta$ whose μ-reach is zero for all $\mu > 0$. However by only considering μ-critical points in an annular regions we can have stability results.

Corollary 3.4. *Let K, L be compact subsets of a manifold with non-negative sectional curvature such that $d_H(K, L) \leq \delta$. Suppose that there are no C-critical points for d_L in the annular region $L_{[a,b]}$. If*

$$C \geq \mu + 2\sqrt{\frac{\delta}{a + \delta}}$$

then there are no μ-critical points for d_K in the annular region $K_{[a+\delta, b-4\delta/(C-\mu)-\delta]}$.

Proof. If x is a μ-critical point with $d_K(x) \in [a + \delta, b - 4\delta/(C - \mu) - \delta]$ then by Proposition 3.3 there exists some C-critical point y with

$$d_L(y) \in [d_L(x), d_L(x) + 4\delta/(C - \mu)] \subset [d_K(x) - \delta, d_K(x) + 4\delta/(C - \mu) + \delta] \subset [a, b]$$

which is a contradiction. $\qquad\square$

Analogous stability results should hold for the cases when $\kappa = 1, -1$. However, we will only later require the case when $\mu = 0$ and so to significantly simplify calculations we restrict to this case.

Proposition 3.5. *Let K, L be compact subsets of $\mathcal{M}$ with $d_H(K, L) \leq \delta$. Let x be a critical point of d_K. Suppose that the sectional curvature of $K_{2d_K(x)}$ is bounded from below by $\kappa = -1$.*
Then for all $C > 0$ there exists a C-critical point y of d_L with $d_L(y) \geq d_L(x)$ and $d(x, y) \leq 4\delta/C$ whenever

$$9\delta \leq 2\tanh(d_K(x))C^2.$$

Proof. Let $C \in (0, 1)$. We want to show that there is some point y such that $\|\nabla_L(y)\| \leq C$ and $d_L(y) \geq d_L(x)$ and $d(x, y) \leq 4\delta/C$. Suppose not. Then there is some $\tilde{\mu} > C$ such that $\|\nabla_L(y)\| \geq \tilde{\mu}$ whenever $d_L(y) \geq d_L(x)$ and $d(x, y) \leq 4\delta/C$

Using $f = d_L - d_L(x)$ and $r = 4\delta/C$ in Lemma 3.2 we know that there exists a point $y \in B(x, 4\delta/C)$ such that $f(y) = 4\delta$; i.e. $d_L(y) = d_L(x) + 4\delta$. From $d_H(K, L) \leq \delta$ we know that $d_K(y) \geq d_K(x) + 2\delta$.

At the same time, Lemma 3.1 implies that $\cosh d_K(y) \leq \cosh d(y, x) \cosh d_K(x)$ and hence $\cosh(d_K(x) + C4\delta/C - 2\delta) \leq \cosh(4\delta/C) \cosh d_K(x)$.

Using the hyperbolic cosh sum formula and dividing through by $\cosh d_K(x)$, this can be rewritten as

$$\cosh(4\delta/C) - \cosh(2\delta) \geq \tanh(d_K(x)) \sinh(2\delta).$$

Our assumption that $9\delta \leq 2\tanh(d_K(x))C^2$ implies that $4\delta/C \leq 8\tanh(d_K(x))C/9 < 1$. Now

$$\cosh(t) - \cosh(2\delta) < \cosh(t) - 1 < \frac{9}{16}t^2$$

whenever $t \in (0, 1)$ and $\sinh(t) > t$ for all t. This means that we can conclude that

$$\frac{9}{16}\left(\frac{4\delta}{C}\right)^2 > \tanh(d_K(x))2\delta$$

and hence that $9\delta > 2\tanh(d_K(x))C^2$. This contradicts our assumption of δ implying that there must exist a suitably nearby C-critical point. $\qquad\square$

4   Reconstruction theorem

Our reconstruction proof will involve finding sufficient conditions for when there exist minimal cone fields. We will use the stability of μ-critical points to show the existence of acute minimal spanning cones of A_δ from points in an annular region.

Lemma 4.1. *Let $C \in (0,1)$ and $R > \delta > 0$ and $\mathcal{M}$ be a manifold with nowhere negative curvature. Let $K \subset \mathcal{M}$ be a compact subset and $x \in K_{R+\delta}$. If there are no C-critical points of d_K in $K_{[d_K(x),d_K(x)+2(R-d_K(x)+\delta)/C]}$ and*

$$\delta \le d_K(x) - \frac{4-C^2}{4+C^2}R$$

there is an acute minimal spanning cone for K_δ from x with length R.

Furthermore, for $a \in (0, R+\epsilon)$, this means that if there are no C-critical points of d_K in $K_{[a,a+2(R+\delta-a)/C]}$ and

$$\delta < a - \frac{4-C^2}{4+C^2}R$$

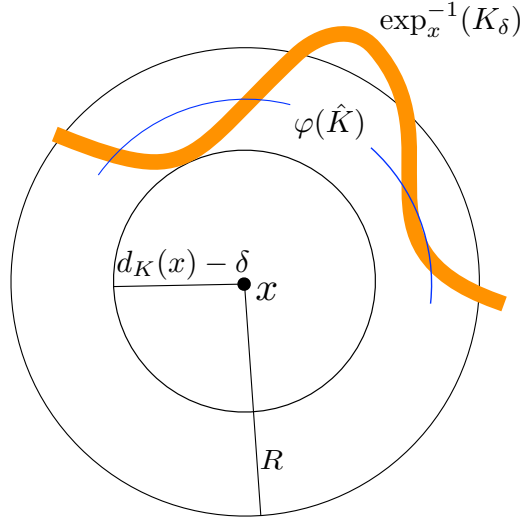*then there is a minimal cone field over $K_{[a,R+\delta]}$ for K_δ with length R.*

Proof. Suppose, by way of contradiction, that 0 is in the convex hull of $\exp_x^{-1} K_\delta \cap B(0,R)$.

Set $\hat{K} := (\exp_x^{-1} K_\delta) \cap B(0,R)$. Define the map

$$\varphi : B(0,R) \to \partial B\left(0, \frac{R+(d_K(x)-\delta)}{2}\right)$$

$$z \mapsto \frac{R+(d_K(x)-\delta)}{2}\frac{z}{\|z\|}.$$

and set $\hat{L}$ to be $\varphi(\hat{K})$. By construction $d_{\hat{L}}(x) = \frac{1}{2}(R+(d_K(x)-\delta))$ and

$$d_H(\exp_x \hat{L}, \exp_x \hat{K}) \le \frac{1}{2}(R-(d_K(x)-\delta)).$$

Construction of $\hat{L} = \varphi(\hat{K})$

Set $L = \exp_x \hat{L} \cup \{a \in K : d(a,x) \ge R\}$. Since taking the union with both sets by the same set can only decrease the Hausdorff distance, $d_H(K,L) \le d_H(\exp_x \hat{K}, \exp_x \hat{L})$. Also, by construction, $d_L(x) = d_{L'}(x) = \frac{1}{2}(R+(d_K(x)-\delta))$.

Since 0 is in the convex hull of $\hat{K}$ (by assumption), there exists $z_1, \ldots, z_m \in \hat{A}$ and $a_1, \ldots a_m > 0$ such that $\Sigma_{i=1}^m a_i z_i = 0$. However this gives

$$\sum_{i=1}^m a_i \frac{2\|z_i\|}{R + d_K(x) - \delta}\varphi(z_i) = 0$$

and hence 0 is in the convex hull of $\hat{L}$. Since all the points in $\hat{L}$ are equidistant from 0, and hence all the points in $\exp_x \hat{L}$ are equidistant to x, we conclude that x is a 0-critical point of $d_{\exp_x \hat{L}}$. Adding points further away from x leaves this quality invariant and hence x is a 0-critical point of d_K.

Our condition that $\delta \le d_K(x) - R(4 - C^2)/(4 + C^2)$ can be rewritten as

$$\frac{1}{2}\left(R - (d_K(x) - \delta)\right) < \frac{1}{2}\left(R + (d_K(x) - \delta)\right)\frac{C^2}{4}.$$

This implies that $d_H(K, L) < d_L(x)C^2/4$, or in other words $C \ge 0 + 2\sqrt{d_H(K,L)/d_L(x)}$. This enables us to apply Proposition 3.3 to say that there exists a C-critical point y of d_K with

$$d_K(y) \in [d_K(x), d_K(x) + 4d_H(K,L)/C] = [d_K(x), d_K(x) + 2(R - d_A(x) + \delta)/C].$$

This contradicts our assumption about the absence of C-critical points in that annular region.

For the second part of the proposition we only need to observe that if $x \in K_{[a, R+\delta]}$ then $[d_K(x), d_K(x) + 2(R - d_K(x) + \delta)/C] \subset [a, a + 2(R - a + \delta)/C]$. $\square$

The process can be applied for the case when $\kappa = -1$ using Proposition 3.5 instead of Proposition 3.3. This leads to the following lemma.

Lemma 4.2. *Let $C \in (0, 1)$ and $R > \delta > 0$ and $\mathcal{M}$ be a manifold with curvature bounded from below by $\kappa = -1$. Let $K \subset \mathcal{M}$ be a compact subset and $x \in K_{R+\delta}$. If there are no C-critical points of d_K in $K_{[d_K(x), d_K(x)+4\delta/C]}$ and*

$$9(R + \delta - d_K(x)) \le 4\tanh\left(\frac{1}{2}(R - \delta + d_K(x)\right)C^2$$

there is an acute minimal spanning cone for K_δ from x with length R.

We observe that so far we have only proved, under certain conditions, the existence of a minimal cone field. Note that the minimal cone field is unlikely to be continuous. What we will need is the existence of a Lipschitz vector field subordinate to it. The importance of minimal cone fields is that vector fields subordinate to them can build deformation retractions.

Lemma 4.3. *Let $0 < \delta < r$, and let K, L be compact subsets of a manifold $\mathcal{M}$ with $d_H(K, L) \le \delta$. Suppose that there exists an acute minimal cone field W over $K_{[r-\delta, r+\delta]}$ for K_δ with parameter r. Let W' be the complementary cone field to W. If X is a Lipschitz vector field strictly subordinate to W', then X induces a deformation retraction from L_r to $K_{r-\delta}$.*

Proof. Since X is a Lipschitz vector field it has a unique continuous integral flow (a standard result, for example [16]). The idea is to follow this flow from each point in L_r until it reaches $K_{r-\delta}$.

Denote the minimal cone field W pointwise by $\{(x, C(w_x, \beta_x))\}$. From Lemma 2.3 we know that any vector sitting strictly inside $C(w_x, \frac{\pi}{2} - \beta_x)$ forms an acute angle when paired with angle vector in $C(w_x, \beta_x)$.

Let X be a vector field strictly subordinate to W'. Let $x \in L_r \cap K_{[r-\delta, r+\delta]}$, and let v be the vector at x in X. Now $x \in B(y, r)$ for some $y \in L$. Let γ_x^y denote a geodesic from x to y of length at

most r, and $\gamma'^y_x(0)$ its tangent vector at x. By construction, $\gamma'^y_x(0) \in C(w_x, \beta_x)$ and hence it forms an acute angle with v. This means that their images form an acute angle in the normal coordinates given by the exponential map at x.

Now consider the normal coordinates given by the exponential map at y. In these coordinates γ^y_x is a radial straight line emitted from the origin. Gauss's lemma (see [17]) tells us that the angle between γ_v and γ^y_x in the normal coordinates at y is acute if and only if the angle between them in the normal coordinates based at x is acute. We have already shown that this second angle is acute.

This means that in the normal coordinates given by the exponential map at y, γ_v must remain inside $B(0, r)$ for some positive amount of time. Since this is true all x we know that the integral flow does not leave L_r. Furthermore, the integral flow of X is always traveling towards K as it lies in W'. For each $x \in K_{[r-\delta, r+\delta]}$, let λ_x be the rate at which the integral flow of X at x is traveling towards K. Since $\lambda_x > 0$ for all $x \in K_{[r-\delta, r+\delta]}$ and $K_{[r-\delta, r+\delta]}$ is compact there is some $\lambda > 0$ which forms a lower bound on how fast the integral flow of X travels towards X.

Construct the deformation retraction from L_r to $K_{r-\delta}$ by following each point along the flow of X until it reaches $K_{r-\delta}$ and then remaining stationary. The uniform lower bound on how fast the integral flow of X travels towards K combined with the observation that every point in L_r is at most 2δ from $K_{r-\delta}$, tells us that in a finite amount of time every point in L_r will be sent to one in $K_{r-\delta}$. $\qquad\square$

Theorem 4.4. *Let $\mu \in (0,1)$, $r > 0$. Let $\mathcal{M}$ be a smooth manifold with nowhere negative curvature such that every point has an injectivity radius greater than r. Let L a compact subset with $d_H(K, L) < \delta$. Suppose that there are no μ-critical points in $K_{[r-\delta, r-\delta+4\delta/\mu]}$ and $(4+\mu^2)\delta < \mu^2 r$. Then L_r deformation retracts to $K_{r-\delta}$.*

Proof. Lemma 4.1 (with $a = r - \delta$ and $R = r$) ensures that there exists a minimal cone field W over $K_{[r-\delta, r+\delta]}$ for K_δ with length r. By Lemma 4.3 it is sufficient to show that there exists a Lipschitz vector field strictly subordinate to the complementary cone W' of W. We will construct a useful cone field which is larger than W which has good continuity properties. For every cone field V such that $V \geq W$ we know that $W' \geq V'$ and hence it will be sufficient to find a Lipschitz vector field strictly subordinate to V'.

Since $(4+\mu^2)\delta < \mu^2 r$ there is some $\epsilon_1 > 0$ such that $(4+\mu^2)\delta < \mu^2 r - 4\epsilon_1$, and hence $R := r + \epsilon_1$ satisfies

$$\delta < (r - \delta - \epsilon_1) - \frac{4 - \mu^2}{4 + \mu^2} R.$$

There also must exist some $\epsilon_2 > 0$ such that the injectivity radius is greater than $R = r + \epsilon_2$. Since there are no μ-critical point in $K_{[r-\delta, r-\delta+4\delta/C]}$ and $\|\nabla_K(x)\|$ is a lower semicontinuous function of x, there must be some $\epsilon_3 > 0$ such that there are no μ-critical points in $K_{[r-\delta-\epsilon_3, r-\delta-\epsilon_3+4(\delta+\epsilon_3)/C]}$. Take ϵ to be the minimum of these ϵ_i. This ϵ will satisfy all three properties.

Since $\mathcal{M}$ is a smooth manifold, it is triangulable. Barycentrically subdivide until we have a triangulation whose simplices have diameter at most ϵ. Let Σ be the simplicial complex by taking the closure of the union of all the simplices whose intersection with $K_{[r-\delta, r+\delta]}$ is non-empty. From the bound on the diameter on the cells in Σ we see that

$$K_{[r-\delta, r+\delta]} \subset \Sigma \subset K_{[r-\delta-\epsilon, r+\delta+\epsilon]}.$$

For each $x \in \Sigma$ let

$$Y(x) := \left(\bigcup_{\{y \in \Delta \mid x \in \Delta\}} B(y, r) \right) \cap K_\delta.$$

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From the restriction on the diameter on any cell in the triangulation and using the triangle inequality we know $B(x,r) \cap K_\delta \subset Y(x) \subset B(x,R) \cap K_\delta$. Given any $x \in \Sigma$ there is some cell Δ containing x such that $\Delta \cap K_{[r-\delta,r+\delta]}$ is nonempty and hence contains some point y. This point y satisfies $B(y,r) \cap K_\delta \neq \emptyset$ which forces $Y(x)$ to be non-empty.

Lemma 4.1 (with $a = r - \delta - \epsilon$ and $R = r + \epsilon$) tells us that for every $x \in K_{[r-\delta-\epsilon,r+\delta+\epsilon]}$ there exists a unique acute minimal spanning cone of length R for K_δ from x. Since $Y(x) \subseteq K_\delta \cap B(x,R)$ and $Y(x)$ is not empty, we conclude that there is also a unique minimal spanning cone $C(u_x, \alpha_x)$ for $Y(x)$ from x of length R.

Set V to be the cone field with attached cones $C(u_x, \alpha_x)$ and V' to be the complementary cone field of V. When we restrict V to $K_{[r-\delta,r+\delta]}$, our construction process ensures $V \geq W$ and hence it is sufficient to find a Lipschitz vector field strictly subordinate to V'.

Given $\Delta \subset \Sigma$, every $x \in \Delta^\circ$ has the same $Y(x)$. Let x_Δ be some point in Δ°. Observe that $Y(x_\Delta)$ is compact. Construct $\hat{V}$ to be the minimal cone field over Δ for $Y(x_\Delta)$ of length R. This minimal cone field exists because for any $y \in \Delta$ we have $B(y,R) \cap Y(x_\Delta) \neq \emptyset$ and $Y(x_\Delta) \subset Y(y)$ (and as observed before there is a minimal cone from y for $Y(y)$ of length R). Proposition 2.2 (with $L = \Delta$ and $K = Y(x)$) tells us that $\hat{V}$ is Lipschitz over Δ. Since V and $\hat{V}$ agree in Δ° we know that V is Lipschitz over Δ°. This implies that V' is also Lipschitz over Δ°.

For every path $\gamma : [0,1] \to \Delta$ with $\gamma(1) \in \partial\Delta$ with $\gamma(t) \in \Delta^\circ$ for $t \in [0,1)$ we have that $\lim_{t \to 1} Y(\gamma(t)) \subseteq Y(\gamma(1))$ which implies that

$$\lim_{t \to 1} C(u_{\gamma(t)}, \pi/2 - \alpha_{\gamma(t)}) \supseteq C(u_{\gamma(1)}, \pi/2 - \alpha_{\gamma(1)}). \tag{4}$$

Informally we could say that that V' has some kind of lower semicontinuity.

We will construct our unit vector field X over Σ which is strictly subordinate to V' inductively over the skeleta of Σ of increasing dimension. First assign to each 0-simplex in Σ any unit vector in the interior of the the cone in V' at that point. Note that this is always possible as the angle of the cone is always positive.

Now suppose, inductively, that we there exists a Lipschitz unit vector field, X_k, over the k - skeleton of Σ which is strictly subordinate to V'. Let Δ be a simplex in Σ of dimension $k+1$. By the inductive hypothesis we have some vector field $X_k|_{\partial\Delta}$ defined over $\partial\Delta$ which is strictly subordinate to V'. We want to extend X_k to a Lipschitz vector field over Δ.

Since V' is Lipschitz on Δ° it has a unique continuous extension V'' over Δ, which is also Lipschitz. From (4), the cone in V' at each point on $\partial\Delta$,is contained in the corresponding cone of V''. This implies that $X_k|_{\partial\Delta}$ is strictly subordinate to $V''|_{\partial\Delta}$. Denote the cone field V'' by $\{(x, C(\tilde{u}_x, \tilde{\alpha}_x))\}$.

Set E to be the fibre bundle over Δ where at each x we assign the subset $C(\tilde{u}_x, \tilde{\alpha}_x)^\circ$ of the unit tangent sphere. We can consider $X_k|_{\partial\Delta}$ as being a section of $E|_{\partial\Delta}$. The fibre bundle E has contractible fibers and a base space with trivial fundamental group so there is no obstruction to extending $X_k|_{\partial\Delta}$ to a continuous section $\tilde{X}_{k+1}$ of E defined over all of Δ.

The space of Lipschitz vector fields which are X_k when restricted to $\partial\Delta$ is dense in the space of continuous vector fields which are X_k when restricted to $\partial\Delta$. Furthermore, continuous sections of E which are X_k when restricted to $\partial\Delta$ form an open subset of continuous vector fields over Δ which are X_k when restricted to $\partial\Delta$. This means there must be a Lipschitz vector field suitably close to $\tilde{X}_{k+1}$ which extends X_k. Set X_{k+1} to be this Lipschitz vector field.

If Δ is a $k+1$ cell then X_{k+1} is strictly subordinate to the corresponding V''. Since V' and V'' agree on Δ° we know that X_{k+1} is strictly subordinate to V'. Thus X_{k+1} is strictly subordinate to V' in the interior of every $k+1$ cell. Since X_{k+1} is also strictly subordinate to V' on the the k-skeleton we can conclude that X_{k+1} is strictly subordinate to V' on the $(k+1)$-skeleton.

Inductively we can construct a Lipschitz vector field X over Σ strictly subordinate to V'. By our construction X is strictly subordinate to W' which allows us to apply Lemma 4.3 to build a defamation retraction from L_r to $K_{r-\delta}$.

$\square$

By doing the same process but using Lemma 4.2 instead of Lemma 4.1 we get the analogous theorem for when the ambient space has its sectional curvature bounded below by -1.

Theorem 4.5. *Let $\mu \in (0,1)$, $r > 0$. Let $\mathcal{M}$ be a smooth manifold whose sectional curvature bounded below by -1 and with an injectivity radius greater than r. Let K and L be compact subsets with $d_H(K,L) < \delta$. Suppose that there are no μ-critical points in $K_{[r-\delta,r-\delta+4\delta/\mu]}$ and $9\delta < 2\tanh(r-\delta)\mu^2$. Then L_r deformations retracts to $K_{r-\delta}$.*

5   Applications to point cloud data

In this section we now consider the situation where we have some unknown compact set A which we are wanting to understand and we can sample A to generate a (potentially noisy) point cloud of A which we will denote by S. Historically conditions geometric conditions have been given on A for when offsets of the S and A are homotopic. This is because A is often assumed to have nice geometric properties whereas S, as a point cloud, has many critical points of its distance function nearby. The corresponding theorem produced using Theorem 4.4 and Theorem 4.5 with $K = A$ and $L = S$ is as follows.

Corollary 5.1. *Let $\mu \in (0,1)$, $r > 0$. Let $\mathcal{M}$ be a smooth manifold with sectional curvature bounded by κ and whose injectivity radius is greater than r. Let A be a compact subset of $\mathcal{M}$ and S be a (potentially noisy) point cloud of A. Suppose that there are no μ-critical points in $A_{[a,b]}$. Then S_r deformation retracts to A_r whenever*

$$d_H(S,A) \le \min\left\{r - a, \frac{b\mu - r\mu}{4 - \mu}\right\}, \ \ and \ d_H(S,A) < \frac{\mu^2 r}{4 + \mu^2} \ if \ \kappa = 0$$

or

$$d_H(S,A) \le \min\left\{r - a, \frac{b\mu - r\mu}{4 - \mu}\right\}, \ \ and \ 9d_H(S,A) < 2\tanh(r - d_H(S,A))\mu^2 \ if \ \kappa = -1.$$

Furthermore if $A_{r-\delta}$ deformation retracts to A then S_r deformation retracts to A.

Proof. In order to apply Theorem 4.4 or Theorem 4.5 we need to make sure that

$$[a,b] \supset [r - d_H(S,A), r - d_H(S,A) + 4d_H(S,A)/\mu]$$

and also that $d_H(S,A) < \frac{\mu^2 r}{4+\mu^2}$ or $9d_H(S,A) < 2\tanh(r - d_H(S,A))\mu^2$ respectively. $\square$

Of general interest is finding the homotopy type of A rather than A_r. However, a sufficient condition for A_b to deformation retract to A_a is that there not be any 0-critical points in $A_b \backslash A_a$ [15]. Also there are many shapes, such as like hairy objects, for which many offsets have a defamation retract even if there are small 0-critical values.

We now want to present a paradigm for finding sufficient conditions on point cloud data for reconstructing any compact subset, lying in any Riemannian manifold, which has positive weak feature size. The first observation we need is that for Corollary 5.1 it is sufficient to have lower

bounds on the sectional curvature and the injectivity radius only for the points in A_{6r} and A_{3r} respectively. This is because no points outside this region are use in any of the proofs. Since A is compact there is some $r > 0$ such that the injectivity radius of every point in A_{3r} is greater than r. Reduce r if necessary to ensure that $r < \mathrm{wfs}(A)$ where $\mathrm{wfs}(A)$ is the weak feature size of A which we have assumed is positive. A_{3r} is compact so there is some finite lower bound on the sectional curvature for points in A_{3r}. By rescaling the metric on the ambient manifold if necessary (and with it scaling r) we can assume that the lower bound on sectional curvature is 0 or -1. This means we can apply Corollary 5.1. It is clear a suitable μ and bound on $d_H(A, S)$ in the Corollary must exist. Because $r < \mathrm{wfs}(A)$ we can further state that the S_r deformation retracts to A. This paradigm of reconstruction processes shows that what the ambient manifold is does not pose a theoretical barrier to the existence of reconstruction proofs.

An alternative approach, as pointed out in [7], is to consider geometric properties of S (or in their case offsets of S) itself rather than A. We can take his approach because S is a compact set and we do not require any smooth structure. This means we can also conclude another corollary with $K = S$ and $L = A$.

Corollary 5.2. *Let $\mu \in (0,1)$, $r > 0$. Let $\mathcal{M}$ be a smooth manifold with sectional curvature bounded by κ whose injectivity radius is greater than r. Let A be a compact subset of $\mathcal{M}$ and S be a (potentially noisy) point cloud of A. Suppose that there are no μ-critical points in $S_{[a,b]}$. Then S_r deformation retracts to A_r whenever*

$$d_H(S, A) \leq \min\left\{ r - a, \frac{b\mu - r\mu}{4 - \mu} \right\}, \ \text{and } d_H(S, A) < \frac{\mu^2 r}{4 + \mu^2} \ \text{if } \kappa = 0$$

or

$$d_H(S, A) \leq \min\left\{ r - a, \frac{b\mu - r\mu}{4 - \mu} \right\}, \ \text{and } d_H(S, A) < \frac{2}{9} \tanh(r - d_H(S, A))\mu^2 \ \text{if } \kappa = -1.$$

Furthermore if A_r is homotopic to A then S_r is homotopic to A.

When the ambient space is Euclidean, it is reasonable to want to compare our reconstruction process to previous ones in the literature. Since these have been quantified in terms of μ-reach we can first compare the required Hausdorff bounds on $\delta := d_H(A, S)$ where A is a compact set with μ-reach $r_\mu > 0$ and S is a point cloud. If we consider the limiting case when $r - \delta + 4\delta/C < r_\mu$ we can apply our reconstruction theorem (in the case of $\kappa = 0$) once $\delta/r_\mu < \mu^2/(4 + 4\mu)$. Notably this is an improvement on the bounds presented in [7], which is $\delta/r_\mu < \mu^2/(5\mu^2 + 12)$, for all μ and an improvement on the bounds in [2], where it is

$$\frac{\delta}{r_\mu} < \frac{-3\mu + 3\mu^2 - 3 + \sqrt{-8\mu^2 + 4\mu^3 + 18\mu + 2\mu^4 + 9 + \mu^6 - 4\mu^5}}{7\mu^2 + 22\mu + \mu^4 - 4\mu^3 + 1},$$

for $\mu < 0.945$.

One advantage of the approach of this paper is not having any requirements about the absence of μ-critical points very close to A. A severe limitation of restricting to set with positive μ-reach is the inability to cope with sets that have cusps. At cusps the the μ-reach is zero for all values of $\mu > 0$. The method used to overcome the shortfalls of μ-reach is to consider offsets of the compact set. For a compact set K, there are no μ-critical points of d_K in $K_{[a,b]}$ if and only if the μ-reach of K_a is at least $b - a$. This means we can compare different reconstruction theorems in terms of a lack of μ critical points in an annular region.

Let us assume that there are no μ-critical points of d_K in $K_{[a,b]}$. For our reconstruction process we need
$$\delta < \min \left\{ \frac{\mu(b-a)}{4}, \frac{\mu^2 b}{4 + 4\mu} \right\}.$$
Here we would use $r = b \frac{4 + \mu^2}{4 + 4\mu}$. In comparison, for the reconstructions in [7] would need
$$\delta < \frac{(b-a)\mu^2}{5\mu^2 + 12}$$
and the the reconstructions in [2] we would need
$$\delta < (b-a) \frac{-3\mu + 3\mu^2 - 3 + \sqrt{-8\mu^2 + 4\mu^3 + 18\mu + 2\mu^4 + 9 + \mu^6 - 4\mu^5}}{7\mu^2 + 22\mu + \mu^4 - 4\mu^3 + 1}$$
which is significantly worse when $b - a$ is small in comparison to b.

One possible future direction is the study of persistent homotopy. Persistent homology has been around for decades as a method of analyzing data [11]. It involves computing the homology classes of different offsets and studying when different homology class are born and die. A similar process could be done for homotopy. For $a < b$ there is an inclusion map $\iota : A_a \to A_b$ which induces maps on the homotopy groups $\iota_* : \pi_k(A_a) \to \pi_k(A_b)$ with changes only when a critical value of the distance function is passed. This paper gives sufficient conditions for $S_{a'}$ and $S_{b'}$ to deformation retract to A_a and A_b respectively, and hence the map induced by inclusion $\pi_k(S_{a'}) \to \pi_k(S_{b'})$ is the same as ι_*.

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